

Gauge theory on kappa-Minkowski revisited: the twist approach

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Abstract. Kappa-Minkowski space-time is an example of noncommutative space-time with potentially interesting phenomenology. However, the construction of field theories on this space is plagued with ambiguities. We propose to resolve certain ambiguities by clarifying the geometrical picture of gauge transformations on the κ -Minkowski space-time in the twist approach. We construct the action for the noncommutative $U(1)$ gauge fields in a geometric way, as an integral of a maximal form. The effective action with the first order corrections in the deformation parameter is obtained using the Seiberg-Witten map to relate noncommutative and commutative degrees of freedom.

1. Introduction and overview

It is generally believed that the picture of space-time as a differentiable manifold should break down at very short distances of the order of the Planck length. There are different proposals for the modified space-time structure which should provide a consistent framework encompassing physics in this regime. These proposals include, among others, the dynamical triangulation as a way of direct geometrical construction of modified space-time, strings and loops as non-local fundamental observables dynamically generating space-time, and a deformation of algebra of functions on a manifold as a way of introducing a 'noncommutative space-time'.

Introducing a non-trivial algebra of coordinates in order to modify the space-time structure can be seen as an attempt to generalise the concept of symmetries that should encompass physics on a quantum manifold. A possible way to describe physics on such a manifold is to construct (effective) field theory compatible with the algebra of coordinates in the framework of deformation quantization. The main advantage of such effective models is that one can extract phenomenological consequences of the space-time modification using standard field-theoretical tools.

In this work our primary interest is to examine the compatibility of the local gauge principle with the deformation of the algebra of functions on a specific example of noncommutative space-time, the κ -Minkowski space-time. The commutation relations of the coordinates of this space-time are of the Lie-algebra type:

$$[\hat{x}^0, \hat{x}^j] = \frac{i}{\kappa} \hat{x}^j, \quad [\hat{x}^i, \hat{x}^j] = 0. \quad (1)$$

One of the interesting properties of this noncommutative space-time is that there is a quantum group symmetry acting on it [1]. It is a dimensionful deformation of the global Poincaré group,

the κ -Poincaré group. The constant κ has dimension of energy and sets a deformation scale. The κ -Minkowski space also provides an arena for formulating new physical concepts. These include the generalisation of Special Relativity with one additional invariant scale known as Doubly Special Relativity [2], and the concept of relativity of locality [3], a proposal in which space-time is observer-dependent projection from the invariant phase space. This makes the κ -Minkowski space-time an example of noncommutative space-time with potentially interesting phenomenology.

We are interested in the construction of the field theory on this space as a step towards extracting observable consequences of underlying noncommutative structure. In our previous work [4] we showed that within the framework defined in [5, 6] one can consistently describe a gauge theory on the κ -Minkowski space-time by explicit construction of $U(1)$ gauge theory coupled to fermions. Although successful, the construction revealed certain ambiguities which were fixed by physical arguments and intuition, rather than by the formalism itself. With this motivation in mind, we use the twist formalism in order to gain a better understanding of the gauge theory on the κ -Minkowski space-time [7]. Note, however, that in this formalism we cannot maintain the κ -Poincaré symmetry; the corresponding symmetry of the twisted κ -Minkowski space is the twisted $igl(1, 3)$ symmetry. One of the advantages of the twist formalism is the straightforward way to define a differential calculus. This enables us to write the action in a geometric way, as an integral of a maximal form. Furthermore, the geometric description we use clearly shows that one cannot decouple translations and gauge symmetries, a generic feature for theories with underlying noncommutative structure. In our model this mixing of internal and space-time symmetries enters in the construction of the (Hodge) dual field strength. We present two different ways of defining the dual field strength leading to the same expanded action in the first order in deformation parameter.

2. Noncommutative spaces from a twist

The main idea of the twist formalism is to first deform the symmetry of the theory and then see the consequences this deformation has on the space-time itself. There is a well defined way to deform the symmetry Hopf algebra. In his paper [8] Drinfel'd introduced a notion of twist. The twist \mathcal{F} is an invertible operator which belongs to $Ug \otimes Ug$, where Ug is the universal enveloping algebra of the symmetry Lie algebra g . The universal enveloping algebra Ug is a Hopf algebra

$$\begin{aligned} [t^a, t^b] &= if^{abc}t^c, \\ \Delta(t^a) &= t^a \otimes 1 + 1 \otimes t^a, \\ \varepsilon(t^a) &= 0, \quad S(t^a) = -t^a. \end{aligned} \tag{2}$$

In the first line t^a label the generators of the symmetry algebra g and the structure constants are labelled by f^{abc} . In the second line the coproduct of the generator t^a is given. It encodes the Leibniz rule and specifies how the symmetry transformation acts on products of fields/representations. In the last line, the counit and the antipode are given. The properties which the twist \mathcal{F} has to satisfy are:

- (i) the cocycle condition

$$(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F}, \tag{3}$$

- (ii) normalization

$$(id \otimes \epsilon)\mathcal{F} = (\epsilon \otimes id)\mathcal{F} = 1 \otimes 1, \tag{4}$$

- (iii) perturbative expansion

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\lambda), \tag{5}$$

where λ is a small deformation parameter. The last property is not necessary. It provides an expansion around the undeformed case in the limit $\lambda \rightarrow 0$. We shall frequently use the notation (sum over $\alpha = 1, 2, \dots \infty$ is understood)

$$\mathcal{F} = f^\alpha \otimes f_\alpha, \quad \mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha, \quad (6)$$

where, for each value of α , \bar{f}^α and \bar{f}_α are two distinct elements of Ug (and similarly f^α and f_α are in Ug). We also introduce the universal \mathcal{R} -matrix

$$\mathcal{R} = \mathcal{F}_{21} \mathcal{F}^{-1}, \quad (7)$$

where by definition $\mathcal{F}_{21} = f_\alpha \otimes f^\alpha$. In the sequel we use the notation

$$\mathcal{R} = R^\alpha \otimes R_\alpha, \quad \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha. \quad (8)$$

The twist acts on the symmetry Hopf algebra and gives the twisted symmetry Hopf algebra

$$\begin{aligned} [t^a, t^b] &= if^{abc}t^c, \\ \Delta_{\mathcal{F}}(t^a) &= \mathcal{F}\Delta(t^a)\mathcal{F}^{-1} \\ \varepsilon(t^a) &= 0, \quad S_{\mathcal{F}}(t^a) = f^\alpha S(f_\alpha)S(t^a)S(\bar{f}^\beta)\bar{f}_\beta. \end{aligned} \quad (9)$$

We see that the algebra remains the same, while in general the comultiplication changes. This leads to the deformed Leibniz rule for the symmetry transformations when acting on product of fields.

We can now use the twist to deform the commutative geometry on space-time (vector fields, one-forms, exterior algebra of forms, tensor algebra). The guiding principle is the observation that every time we have a bilinear map

$$\mu : X \times Y \rightarrow Z,$$

where X, Y, Z are vector spaces and when there is an action of the Lie algebra g (and therefore of \mathcal{F}^{-1}) on X and Y we can combine the map μ with the action of the twist. In this way we obtain the deformed map μ_\star :

$$\mu_\star = \mu \mathcal{F}^{-1}. \quad (10)$$

The cocycle condition (3) implies that if μ is an associative product then also μ_\star is an associative product.

Let us analyze this deformation in more detail. For convenience we now consider one particular class of twists, the Abelian twists

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b}. \quad (11)$$

Here θ^{ab} is a constant antisymmetric matrix and $X_a = X_a^\mu \partial_\mu$ are commuting vector fields. The algebra of vector fields on the space-time M we label with Ξ and the universal enveloping algebra of this algebra with $U\Xi$. Then \mathcal{F} belongs to $U\Xi \otimes U\Xi$. In the view of (2)-(5), the symmetry algebra is the algebra of diffeomorphisms generated by vector fields $\xi = \xi^\mu \partial_\mu \in \Xi$. Note that depending on the choice of vector fields X_a one can also consider a subalgebra of the diffeomorphism algebra such as Poincaré or conformal algebra.

Applying the inverse of the twist (11) to the usual point-wise multiplication of functions on the space-time M , $\mu(f \otimes g) = f \cdot g$, we obtain the \star -product of functions

$$\begin{aligned} f \star g &= \mu \mathcal{F}^{-1}(f \otimes g) \\ &= \bar{f}^\alpha(f) \bar{f}_\alpha(g) \\ &= \bar{R}^\alpha(g) \star \bar{R}_\alpha(f). \end{aligned} \quad (12)$$

We see that the R -matrix encodes the noncommutativity of the \star -product. The action of the twist (\bar{f}^α and \bar{f}_α) on the functions f and g is via the Lie derivative.

The product between functions and one-forms is given by following the general prescription

$$h \star \omega = \bar{f}^\alpha(h) \bar{f}_\alpha(\omega) \quad (13)$$

with an arbitrary one-form ω . The action of \bar{f}_α on forms is given via the Lie derivative. Functions can be multiplied from the left or from the right,

$$h \star \omega = \bar{f}^\alpha(h) \bar{f}_\alpha(\omega) = \bar{R}^\alpha(\omega) \star \bar{R}_\alpha(h). \quad (14)$$

Exterior forms form an algebra with the wedge product $\wedge : \Omega^\bullet \times \Omega^\bullet \rightarrow \Omega^\bullet$. We \star -deform the wedge product on two arbitrary forms ω and ω' into the \star -wedge product,

$$\omega \wedge_\star \omega' = \bar{f}^\alpha(\omega) \wedge \bar{f}_\alpha(\omega'). \quad (15)$$

We denote by Ω_\star^\bullet the linear space of forms equipped with the \star -wedge product \wedge_\star .

As in the commutative case exterior forms are totally \star -antisymmetric (contravariant) tensor-fields. For example, two-form $\omega \wedge_\star \omega'$ is the \star -antisymmetric combination

$$\begin{aligned} \omega \wedge_\star \omega' &= \bar{f}^\alpha(\omega) \wedge \bar{f}_\alpha(\omega') \\ &= \omega \otimes_\star \omega' - \bar{R}^\alpha(\omega') \otimes_\star \bar{R}_\alpha(\omega), \\ &= -\bar{R}^\alpha(\omega') \wedge_\star \bar{R}_\alpha(\omega), \end{aligned} \quad (16)$$

with the \star -tensor product defined as

$$T_1 \otimes_\star T_2 = \bar{f}^\alpha(T_1) \otimes \bar{f}_\alpha(T_2). \quad (17)$$

The usual exterior derivative $d : \mathcal{A}_x \rightarrow \Omega$, as it commutes with the Lie derivative, satisfies the Leibniz rule $d(f \star g) = df \star g + f \star dg$ and is therefore also the \star -exterior derivative. One can rewrite the usual exterior derivative of a function using the \star -product as

$$\begin{aligned} df &= (\partial_\mu f) dx^\mu \\ &= (\partial_\mu^\star f) \star dx^\mu, \end{aligned} \quad (18)$$

where the new derivatives ∂_μ^\star are defined by this equation.

The usual integral is cyclic under the \star -exterior products of forms, that is up to boundary terms we have

$$\int \omega_1 \wedge_\star \omega_2 = (-1)^{d_1 \cdot d_2} \int \omega_2 \wedge_\star \omega_1, \quad (19)$$

where $d = \deg(\omega)$, $d_1 + d_2 = m$ and m is the dimension of the space-time M . This property holds for the Abelian twist (11). More generally, one can show [9] that this property holds for any twist that satisfies the condition $S(\bar{f}^\alpha) \bar{f}_\alpha = 1$, with the antipode S .

3. Kappa-Minkowski via twist

Algebraically, the four-dimensional κ -Minkowski space-time can be introduced as a quotient of the algebra freely generated by coordinates \hat{x}^μ divided by the ideal generated by the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = i C_\rho^{\mu\nu} \hat{x}^\rho, \quad \mu, \nu, \rho = 0, \dots, 3. \quad (20)$$

Defining

$$C_\rho^{\mu\nu} = a(\delta_0^\mu \delta_\rho^\nu - \delta_0^\nu \delta_\rho^\mu) \quad (21)$$

the commutation relations (20) can be rewritten as

$$[\hat{x}^0, \hat{x}^j] = ia\hat{x}^j, \quad [\hat{x}^i, \hat{x}^j] = 0. \quad (22)$$

The metric of the κ -Minkowski space-time is $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The deformation parameter a is related to the frequently used parameter κ as $a = 1/\kappa$. Latin indices denote the space dimensions, zero the time dimension and the Greek indices refer to all dimensions.

The choice of twist is not unique and it depends on the properties that we want to obtain/preserve. We choose the following twist

$$\begin{aligned} \mathcal{F} &= e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b} \\ &= e^{-\frac{ia}{2}(\partial_0 \otimes x^j \partial_j - x^j \partial_j \otimes \partial_0)}, \end{aligned} \quad (23)$$

with two commuting vector fields $X_1 = \partial_0$ and $X_2 = x^j \partial_j$ and

$$\theta^{ab} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}. \quad (24)$$

Our choice is motivated by the fact that the twist (23) leads (see below) to the hermitean \star -product and the cyclic integral, the properties crucial for the construction of an action. One can check that this twist fulfils the conditions (3), (4) and (5) with the small deformation parameter $\lambda = a$. Deformed symmetry concerned, note that X_2 is not in the universal enveloping algebra of the Poincaré algebra. Therefore we have to enlarge the Poincaré algebra $iso(1,3)$ to the inhomogeneous general linear algebra $igl(1,3)$ and twist this algebra instead of $iso(1,3)$. The generators (given in the representation on the space of functions/fields) and the commutation relations of $igl(1,3)$ are

$$\begin{aligned} M_{\mu\nu} &= x_\mu \partial_\nu, \quad P_\mu = \partial_\mu, \\ [P_\mu, P_\nu] &= 0, \quad [M_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\rho\nu}. \end{aligned} \quad (25)$$

Let us discuss the consequences of the twist (23). The action of the twist (23) on the $igl(1,3)$ algebra follows from (9) and it has been analysed in detail in [10]. Here we just summarise the most important results. The algebra (25) remains the same. On the other hand, since $X_2 = x^j \partial_j$ does not commute with the generators ∂_μ and $M_{\mu\nu}$ the comultiplication and the antipode change. Here we just give the result for the twisted comultiplication, the other results can be found in [10],

$$\begin{aligned} \Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta P_j &= P_j \otimes e^{-\frac{i}{2}aP_0} + e^{\frac{i}{2}aP_0} \otimes P_j, \\ \Delta M_{ij} &= M_{ij} \otimes 1 + 1 \otimes M_{ij}, \\ \Delta M_{0j} &= M_{0j} \otimes e^{-\frac{i}{2}aP_0} + e^{\frac{i}{2}aP_0} \otimes M_{0j} - \frac{i}{2}aP_j \otimes D + \frac{i}{2}aD \otimes P_j, \\ \Delta M_{j0} &= M_{j0} \otimes e^{-\frac{i}{2}aP_0} + e^{\frac{i}{2}aP_0} \otimes M_{j0}, \\ \Delta M_{00} &= M_{00} \otimes 1 + 1 \otimes M_{00} - \frac{i}{2}aP_0 \otimes D + \frac{i}{2}aD \otimes P_0. \end{aligned} \quad (26)$$

We introduced the notation $D = x^j \partial_j$. Note that κ -Poincaré symmetry found in [1] will not be a symmetry of our twisted κ -Minkowski space. The corresponding symmetry of the twisted κ -Minkowski space is the twisted $igl(1,3)$ symmetry. The twisted symmetry does not have

the usual dynamical significance and there is no Noether procedure associated with it. We view this symmetry as a way of bookkeeping, a prescription that allow us to consistently apply deformation in the theory.

The inverse of the twist (23) defines the \star -product between functions/fields on the κ -Minkowski space-time

$$\begin{aligned} f \star g &= \mu_\star \{f \otimes g\} \\ &= \mu \{ \mathcal{F}^{-1} f \otimes g \} \end{aligned} \quad (27)$$

$$\begin{aligned} &= \mu \{ e^{\frac{ia}{2}(\partial_0 \otimes x^j \partial_j - x^j \partial_j \otimes \partial_0)} f \otimes g \} \\ &= f \cdot g + \frac{ia}{2} x^j ((\partial_0 f) \partial_j g - (\partial_j f) \partial_0 g) + \mathcal{O}(a^2) \\ &= f \cdot g + \frac{i}{2} C_\lambda^{\rho\sigma} x^\lambda (\partial_\rho f) \cdot (\partial_\sigma g) + \mathcal{O}(a^2), \end{aligned} \quad (28)$$

with $C_\lambda^{\rho\sigma}$ given in (21). This product is associative, noncommutative and hermitean

$$\overline{f \star g} = \bar{g} \star \bar{f}. \quad (29)$$

The usual complex conjugation we label with “bar”. In the zeroth order (28) reduces to the usual point-wise multiplication. Of course, we obtain

$$[x^0 \star, x^j] = x^0 \star x^j - x^j \star x^0 = i a x^j, \quad [x^i \star, x^j] = 0. \quad (30)$$

One of the advantages of the twist formalism is the straightforward way to define a differential calculus. Namely, as said in the previous section, we just adopt the undeformed differential calculus with the following properties

$$\begin{aligned} d(f \star g) &= df \star g + f \star dg, \\ d^2 &= 0, \\ df &= (\partial_\mu f) dx^\mu = (\partial_\mu^\star f) \star dx^\mu. \end{aligned} \quad (31)$$

The basis one-forms are dx^μ . Knowing that the action of a vector field on a form is given via Lie derivative one can show that

$$X_1(dx^\mu) = 0, \quad X_2(dx^\mu) = \delta_j^\mu dx^j. \quad (32)$$

Using these relations one obtains that the basis one-forms anticommute but do not \star -commute with functions. Instead they fulfil

$$\begin{aligned} dx^\mu \wedge_\star dx^\nu &= dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu = -dx^\nu \wedge_\star dx^\mu, \\ f \star dx^0 &= dx^0 \star f, \quad f \star dx^j = dx^j \star e^{ia\partial_0} f. \end{aligned} \quad (33)$$

Arbitrary one-forms $\omega_1 = \omega_{1\mu} \star dx^\mu$ and $\omega_2 = \omega_{2\mu} \star dx^\mu$ do not anticommute

$$\omega_1 \wedge_\star \omega_2 = -\bar{R}^\alpha(\omega_2) \wedge_\star \bar{R}_\alpha(\omega_1), \quad (34)$$

where the inverse of the \mathcal{R} matrix is given by

$$\mathcal{R}^{-1} = \mathcal{F}^2 = e^{-ia(\partial_0 \otimes x^j \partial_j - x^j \partial_j \otimes \partial_0)}. \quad (35)$$

The \star -derivatives follow from (31) and are given by

$$\begin{aligned}\partial_0^\star &= \partial_0, & \partial_j^\star &= e^{-\frac{i}{2}a\partial_0}\partial_j, \\ \partial_0^\star(f \star g) &= (\partial_0^\star f) \star g + f \star (\partial_0^\star g), \\ \partial_j^\star(f \star g) &= (\partial_j^\star f) \star e^{-ia\partial_0}g + f \star (\partial_j^\star g).\end{aligned}\tag{36}$$

The usual integral of a maximal form is cyclic

$$\int \omega_1 \wedge_\star \omega_2 = (-1)^{d_1 \cdot d_2} \int \omega_2 \wedge_\star \omega_1,\tag{37}$$

with $d_1 + d_2 = 4$. Since basis one-forms anticommute the volume form remains undeformed

$$d_\star^4 x := dx^0 \wedge_\star dx^1 \wedge_\star \dots dx^3 = dx^0 \wedge dx^1 \wedge \dots dx^3 = d^4 x.\tag{38}$$

4. $U(1)$ gauge theory

In this section we formulate pure noncommutative $U(1)$ gauge theory on the twisted κ -Minkowski space-time. The coupling to matter was analysed in [7]. We start by introducing the noncommutative connection

$$A = A_\mu \star dx^\mu,$$

written in the coordinate basis. The transformation law of the noncommutative connection is given by

$$\delta_\alpha^\star A = d\Lambda_\alpha + i[\Lambda_\alpha \star A],\tag{39}$$

or in the components

$$\delta_\alpha^\star A_0 = \partial_0 \Lambda_\alpha + i[\Lambda_\alpha \star A_0],\tag{40}$$

$$\delta_\alpha^\star A_j = \partial_j^\star \Lambda_\alpha + i\Lambda_\alpha \star A_j - iA_j \star e^{-ia\partial_0} \Lambda_\alpha.\tag{41}$$

The field-strength tensor is a two-form given by

$$F = \frac{1}{2} F_{\mu\nu} \star dx^\mu \wedge_\star dx^\nu = dA - iA \wedge_\star A,\tag{42}$$

or in components

$$F_{0j} = \partial_0^\star A_j - \partial_j^\star A_0 - iA_0 \star A_j + iA_j \star e^{-ia\partial_0} A_0,\tag{43}$$

$$F_{ij} = \partial_i^\star A_j - \partial_j^\star A_i - iA_i \star e^{-ia\partial_0} A_j + iA_j \star e^{-ia\partial_0} A_i.\tag{44}$$

One can check that (42) transforms covariantly,

$$\delta_\alpha^\star F = i[\Lambda_\alpha \star F].\tag{45}$$

As a next step we would like to construct the action. In the undeformed, commutative gauge theory¹ one writes the action for the gauge field using the Hodge dual of the field-strength tensor $\star F^{(0)}$

$$\begin{aligned}S^{(0)} &= -\frac{1}{2} \int F^{(0)} \wedge (\star F^{(0)}), \\ \star F^{(0)} &= \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{(0)\alpha\beta} dx^\mu \wedge dx^\nu.\end{aligned}$$

¹ In the following, the undeformed, commutative fields will be denoted by superscript (0), e.g., $F_{\alpha\beta}^{(0)} = \partial_\alpha A_\beta^{(0)} - \partial_\beta A_\alpha^{(0)}$, is the usual $U(1)$ field strength.

The indices on $F^{(0)\alpha\beta}$ are raised with the flat metric $\eta_{\mu\nu}$ and

$$\delta_\alpha(*F^{(0)}) = i[\alpha, *F^{(0)}] = 0 \quad (46)$$

since we work with $U(1)$ gauge theory.

We try to generalise this to the κ -Minkowski space-time. We write the noncommutative gauge field action as

$$S = c_1 \int F \wedge_\star (*F), \quad (47)$$

where $*F$ is the noncommutative Hodge dual field strength. In order to have an action invariant under the noncommutative gauge transformations (39) the dual field strength has to transform covariantly

$$\delta_\alpha^*(F) = i[\Lambda_\alpha \star, *F]. \quad (48)$$

The obvious guess for the noncommutative Hodge dual

$$*F = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \star dx^\mu \wedge_\star dx^\nu \quad (49)$$

does not work since it does not transform covariantly

$$\delta_\alpha^*(F) = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} (\delta_\alpha^* F^{\alpha\beta}) \star dx^\mu \wedge_\star dx^\nu \neq i[\Lambda_\alpha \star, *F]. \quad (50)$$

Therefore we have to try something else. We assume that $*F$ has the form

$$*F := \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} X^{\alpha\beta} \star dx^\mu \wedge_\star dx^\nu, \quad (51)$$

where $X^{\alpha\beta}$ are unknown components that should be determined from the condition (48). Unfortunately, we were unable to find consistent Ansatz for $X^{\alpha\beta}(A_\mu)$ in the closed form. Up to the first order in the deformation parameter we find:

$$\begin{aligned} X^{0j} &= F^{0j} - a A_0 \star F^{0j}, \\ X^{jk} &= F^{jk} + a A_0 \star F^{jk}. \end{aligned} \quad (52)$$

Inserting this into (51) gives dual field strength that does transform covariantly under the gauge transformations.

Going back to the action (47) and writing it more explicitly we obtain

$$S = -\frac{1}{4} \int \left\{ 2F_{0j} \star e^{-ia\partial_0} X^{0j} + F_{ij} \star e^{-2ia\partial_0} X^{ij} \right\} \star d^4x. \quad (53)$$

where the components of F and X are given in (43), (44) and (52). The terms $e^{-ia\partial_0} X^{0j}$ and $e^{-2ia\partial_0} X^{ij}$ come from \star -commuting the basis one-forms with the components $X^{\mu\nu}$. The constant c_1 is fixed in such a way as to give the good commutative limit of the action (53).

5. Seiberg-Witten map

Next, we would like to extract from the action (53) the first order corrections in the deformation parameter. To this end we use the Seiberg-Witten (SW) map [11, 12, 13]. The idea behind the SW map is that the noncommutative gauge transformation is induced by the commutative one, $\delta_\alpha \rightarrow \delta_\alpha^*$. This means that we can express the noncommutative fields and gauge parameter as functions of the commutative ones, e.g., $\Lambda_\alpha = \Lambda_\alpha(A_\mu^{(0)})$, $A_\mu = A_\mu(A_\mu^{(0)})$. In this way the

number of degrees of freedom in the noncommutative theory reduces to the number of degrees of freedom of the corresponding commutative theory. Demanding that the algebra of gauge transformations, defined in (39), closes gives the consistency condition:

$$(\delta_\alpha^\star \delta_\beta^\star - \delta_\beta^\star \delta_\alpha^\star)A = \delta_{-i[\alpha, \beta]}^\star A. \quad (54)$$

This condition is taken as an equation giving the expression for noncommutative gauge parameter Λ_α in terms of the commutative gauge parameter α and the commutative gauge field $A_\mu^{(0)}$. In general, the solution can not be obtained in the closed form. Expanding the gauge parameter Λ_α in the orders of the deformation parameter enables one to solve the equation order by order. In this paper we are interested in the first order correction, therefore we construct the SW map only up to first order. Writing Λ_α as

$$\Lambda_\alpha = \alpha + \Lambda_\alpha^1 + \dots + \Lambda_\alpha^k + \dots, \quad (55)$$

assuming $\Lambda_\alpha^k = \Lambda_\alpha^k(A_\mu^{(0)})$, and expanding the \star -product in the equation (54), we obtain the inhomogeneous equation for Λ_α^1 :

$$\delta_\alpha \Lambda_\beta^1 - \delta_\beta \Lambda_\alpha^1 = -C_\lambda^{\rho\sigma} x^\lambda (\partial_\rho \alpha) (\partial_\sigma \beta). \quad (56)$$

Note that $\delta_\alpha \Lambda_\beta^1 \neq 0$ since Λ_β^1 is a function of the commutative gauge parameter β and the commutative gauge field $A_\mu^{(0)}$ and $\delta_\alpha A_\mu^{(0)} = \partial_\mu \alpha \neq 0$. The solution of equation (56) is given by

$$\Lambda_\alpha^1 = -\frac{1}{2} C_\lambda^{\rho\sigma} x^\lambda A_\rho^{(0)} \partial_\sigma \alpha. \quad (57)$$

This solution is not unique, one can always add a solution of the homogeneous equation to it. This is the freedom in the SW map. In the case of $U(1)$ gauge group the only homogeneous term is of the form

$$\Lambda_\alpha^{\text{hom}} = b_1 C_\lambda^{\rho\sigma} x^\lambda F_{\rho\sigma}^{(0)} \alpha. \quad (58)$$

However, this term does not lead to a solvable equation for the noncommutative gauge field and therefore we shall not consider it. The noncommutative gauge parameter up to first order in the deformation parameter reads

$$\Lambda_\alpha = \alpha - \frac{1}{2} C_\lambda^{\mu\nu} x^\lambda A_\mu^{(0)} \partial_\nu \alpha. \quad (59)$$

In order to find the SW map for gauge field $A_\mu(A_\mu^{(0)})$, we assume that $A_\mu = A_\mu^0 + A_\mu^1 + \dots$, insert this expansion into (40) and (41), and expand the \star -product. Thus obtained equation for $A_\mu(A_\mu^{(0)})$ we solve up to the first order we obtain:

$$\begin{aligned} A_\mu = & A_\mu^{(0)} - \frac{a}{2} \delta_\mu^j \left(i \partial_0 A_j^{(0)} + A_0^{(0)} A_j^{(0)} \right) + \frac{1}{2} C_\lambda^{\rho\sigma} x^\lambda \left(F_{\rho\mu}^{(0)} A_\sigma^{(0)} - A_\rho^{(0)} \partial_\sigma A_\mu^{(0)} \right) \\ & + d_1 C_\lambda^{\rho\sigma} x^\lambda \partial_\rho F_{\sigma\mu}^{(0)} + d_2 a F_{\mu 0}^{(0)}. \end{aligned} \quad (60)$$

The terms with the real undetermined coefficients d_1 and d_2 are the solutions of the homogeneous equation and represent the freedom of the SW map. Note that the connection one-form A is real, but the components A_μ are not necessarily real due to the \star -product in $A = A_\mu \star dx^\mu$.

Inserting the solution (60) into (43) and (44) results in the SW map for the field strength tensor:

$$F_{0j} = F_{0j}^{(0)} - \frac{ia}{2}\partial_0 F_{0j}^{(0)} - aA_0^{(0)}F_{0j}^{(0)} + C_\lambda^{\rho\sigma}x^\lambda \left(F_{\rho 0}^{(0)}F_{\sigma j}^{(0)} - A_\rho^{(0)}\partial_\sigma F_{0j}^{(0)} \right) + a(d_1 - d_2)\partial_0 F_{0j}^{(0)}, \quad (61)$$

$$F_{ij} = F_{ij}^{(0)} - ia\partial_0 F_{ij}^{(0)} - 2aA_0^{(0)}F_{ij}^{(0)} + C_\lambda^{\rho\sigma}x^\lambda \left(F_{\rho i}^{(0)}F_{\sigma j}^{(0)} - A_\rho^{(0)}\partial_\sigma F_{ij}^{(0)} \right) + a(d_1 - d_2)\partial_0 F_{ij}^{(0)}. \quad (62)$$

6. Equations of motion and expanded action

Having defined the action and the SW map for the gauge fields we are ready to calculate the equations of motion for the fields. We expand the action (53) in the first order in the deformation parameter using the SW map and expanding the \star -product. This gives an effective action for the undeformed gauge fields with the first order corrections coming from the deformation we introduced:

$$S^{\text{eff}} = -\frac{1}{4} \int d^4x \left\{ F_{\mu\nu}^{(0)} F^{(0)\mu\nu} - \frac{1}{2} C_\lambda^{\rho\sigma} x^\lambda F^{(0)\mu\nu} F_{\mu\nu}^{(0)} F_{\rho\sigma}^{(0)} + 2C_\lambda^{\rho\sigma} x^\lambda F^{(0)\mu\nu} F_{\mu\rho}^{(0)} F_{\nu\sigma}^{(0)} \right\}. \quad (63)$$

Note that there are no ambiguous terms in the expanded action coming from the freedom in the SW map; all such terms turned out to be total derivative terms and therefore they dropped out from the expanded action. The equation of motion for the gauge field is:

$$\partial_\mu F^{(0)\alpha\mu} = -\frac{a}{4}\delta_0^\alpha F^{(0)\mu\nu} F_{\mu\nu}^{(0)} + 2aF^{(0)\alpha\mu} F_{0\mu}^{(0)} + C_\lambda^{\rho\sigma} x^\lambda \left(F_\rho^{(0)\mu} \partial_\mu F_\sigma^{(0)\alpha} + F_{\mu\sigma}^{(0)} \partial_\rho F^{(0)\mu\alpha} \right). \quad (64)$$

We see that there is no modification of the dispersion relation for the free photon field $A_\mu^{(0)}$ in the first order of the deformation parameter. And this result does agree with our previous findings, see analysis in [14]. However, the x -dependent terms in our expanded action clearly demand better understanding, possibly in terms of geometric degrees of freedom. Furthermore, one needs to understand the renormalization properties of the theory before making any predictions. Based on the results obtained in field theory on the canonically deformed space-time one does expect additional terms in the action which render theory renormalizable [15]. Finally, the second order corrections in the deformation parameter might turn out to be essential for deforming of the dispersion relations.

7. $U(1)$ gauge theory - take two

We introduced the noncommutative $U(1)$ gauge theory in Section 4, where we have chosen to work in the coordinate basis. The principal advantage of doing the explicit calculations in the coordinate basis is that one works with the flat metric. However, the fact that the basis one-forms dx_μ do not commute with functions (33) presented a (technical) obstruction in the construction of the gauge-covariant expression for the Hodge dual field strength. In this section we discuss the construction of the noncommutative $U(1)$ gauge theory using so-called natural/nice/central basis [16] for the explicit calculations. The main advantage of working in this basis is that the basis one-forms do commute with functions.

We change from the coordinate basis

$$x^\mu = (t = x^0, x, y, z), \quad dx^\mu = (dt, dx, dy, dz), \quad \partial_\mu = (\partial_t, \partial_x, \partial_y, \partial_z)$$

to the nice basis which is given by

$$x^a = (t, r, \theta, \varphi), \quad \theta^a = (dt, \frac{dr}{r}, d\theta, d\varphi), \quad e_a = (\partial_t, r\partial_r, \partial_\theta, \partial_\varphi). \quad (65)$$

We rewrite the twist (28) in the new basis as

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b} = e^{-\frac{ia}{2}(\partial_t \otimes r\partial_r - r\partial_r \otimes \partial_t)} \quad (66)$$

with $X_1 = \partial_t = e_0$ and $X_2 = x^j \partial_j = r\partial_r = e_1$. Consequently, the \star -product between the functions is now given as:

$$\begin{aligned} f \star g &= \mu \mathcal{F}^{-1}(f \otimes g) \\ &= f \cdot g + \frac{ia}{2}((e_0 f)(e_1 g) - (e_1 f)(e_0 g)) + \mathcal{O}(a^2) \\ &= f \cdot g + \frac{ia}{2}((\partial_t f)(r\partial_r g) - (r\partial_r f)(\partial_t g)) + \mathcal{O}(a^2). \end{aligned} \quad (67)$$

As we already mention, the \star -product between the functions and the basis one-forms θ^a is trivial

$$f \star \theta^a = \theta^a \star f = f \cdot \theta^a. \quad (68)$$

This is a consequence of the fact that the Lie-derivatives along the vector fields X_1 and X_2 commute with the basis one-forms; $\mathcal{L}_{e_0}\theta^a = \mathcal{L}_{e_1}\theta^a = 0$. However, the new basis is not flat, and the metric is given by $g_{ab} = \text{diag}(1, -r^2, -r^2, -r^2 \sin^2 \theta)$. The twist (66) is semi-Killing since the metric does not depend on t . This in particular means that the \star -inverse of the metric tensor g_{ab} is the same as the usual inverse $g_{ab} \star g^{ac} = g_{ab} g^{ac} = \delta_a^c$. The volume element is

$$d^4x = \sqrt{-g} \epsilon_{abcd} \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d = r^2 \sin \theta dt dr d\theta d\varphi. \quad (69)$$

The gauge field or the connection $A = A_a \theta^a$ is one-form and under the infinitesimal noncommutative gauge transformations it transforms as

$$\begin{aligned} \delta_\alpha^\star A &= d\Lambda_\alpha + i[\Lambda_\alpha \star A] \\ \delta_\alpha^\star A_a &= e_a \Lambda_\alpha + i[\Lambda_\alpha \star A_a]. \end{aligned} \quad (70)$$

The last line follows from (68). The field-strength tensor is defined as usual and it transforms covariantly under the noncommutative gauge transformations

$$\begin{aligned} F &= dA - iA \wedge_\star A, \quad F_{ab} = e_a A_b - e_b A_a - i[A_a \star A_b] \\ \delta_\alpha^\star F &= i[\Lambda_\alpha \star F], \quad \delta_\alpha^\star F_{ab} = i[\Lambda_\alpha \star F_{ab}]. \end{aligned} \quad (71)$$

The Hodge dual field-strength tensor we define generalizing the usual expression for the Hodge dual in curved space given by

$$\star F^{(0)} = \frac{1}{2} \epsilon_{abcd} \sqrt{-g} g^{ae} g^{bf} F_{ef}^{(0)} \theta^c \wedge \theta^d. \quad (72)$$

Since we want that $\star F$ transforms covariantly under the noncommutative gauge transformations we have to covariantize the metric. More precisely, we have to covariantize the whole expression $\sqrt{-g} g^{ae} g^{bf}$. Let us define

$$\star F = \frac{1}{2} \epsilon_{abcd} G^{abcf} \star F_{ef} \star \theta^c \wedge_\star \theta^d. \quad (73)$$

Here G^{abcf} is the quantity that under noncommutative gauge transformations transforms covariantly

$$\delta_\alpha^\star G^{abcf} = i[\Lambda_\alpha \star G^{abcf}], \quad (74)$$

and in the limit $a \rightarrow 0$ it reduces to $\sqrt{-g}g^{ae}g^{bf}$. Using the Seiberg-Witten map for Λ_α rewritten in the new basis

$$\Lambda_\alpha = \alpha - \frac{a}{2}(A_0^{(0)}(e_1\alpha) - A_1^{(0)}(e_0\alpha)), \quad (75)$$

and expanding the \star -products in equation (74) the solution for G^{aebf} up to first order in a follows

$$G^{aebf} = \sqrt{-g}g^{ae}g^{bf} - aA_0^{(0)}e_1(\sqrt{-g}g^{ae}g^{bf}). \quad (76)$$

For completeness, let us write the Seiberg-Witten map solutions for A and F in the nice basis:

$$A_a = A_a^{(0)} + \frac{a}{2}\left(A_1^{(0)}F_{0a}^{(0)} - A_0^{(0)}F_{1a}^{(0)} + A_1^{(0)}(e_0A_a^{(0)}) - A_0^{(0)}(e_1A_a^{(0)})\right), \quad (77)$$

$$F_{ab} = F_{ab}^{(0)} + a\left(F_{0a}^{(0)}F_{1b}^{(0)} - F_{1a}^{(0)}F_{0b}^{(0)} - A_0^{(0)}(e_1F_{ab}^{(0)}) + A_1^{(0)}(e_0F_{ab}^{(0)})\right). \quad (78)$$

Having all these results at hand, we write the gauge invariant action for pure $U(1)$ gauge theory on κ -Minkowski as:

$$S_g = -\frac{1}{2} \int (*F) \wedge_\star F, \quad (79)$$

with F and $*F$ given by (78) and (73) respectively. Expanding this action up to first order in a we obtain

$$S = -\frac{1}{4} \int d^4x \left\{ F_{ab}^{(0)} F^{(0)ab} + a F^{(0)ab} (4F_{0a}^{(0)} F_{1b}^{(0)} - F_{01}^{(0)} F_{ab}^{(0)}) \right\}, \quad (80)$$

with $d^4x = r^2 \sin\theta dt dr d\theta d\varphi$. We used that

$$\begin{aligned} \int (\partial_r f) dt dr d\theta d\varphi &= \int (r \partial_r f) dt \frac{dr}{r} d\theta d\varphi = \text{surface term} = 0, \\ \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d &= \epsilon^{abcd} dt \wedge \frac{dr}{r} \wedge d\theta \wedge d\varphi. \end{aligned} \quad (81)$$

Note that the action (80) has the same form as the first order expanded action for the noncommutative gauge field in the case of θ -constant deformation. This is the consequence of the particular choice of basis in which the twist looks like the Moyal-Weyl twist. Note that the change of coordinates (65) cannot be done globally since it is not well defined in the origin $x^\mu = 0$.

Finally we would like to compare the result (80) with the expanded action we obtained in the coordinate basis. The change from one basis to the other is done via the matrix L , for example

$$dx^\mu = L^\mu_a \theta^a. \quad (82)$$

We find

$$L^\mu_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & \frac{xz}{\sqrt{x^2+y^2}} & -y \\ 0 & y & \frac{yz}{\sqrt{x^2+y^2}} & x \\ 0 & z & -\sqrt{x^2+y^2} & 0 \end{pmatrix}. \quad (83)$$

Then

$$\begin{aligned} F^{(0)ab} F_{0a}^{(0)} F_{1b}^{(0)} &= x^j F^{(0)\mu\nu} F_{0\mu}^{(0)} F_{j\nu}^{(0)}, \\ F^{(0)ab} F_{01}^{(0)} F_{ab}^{(0)} &= x^j F^{(0)\mu\nu} F_{0j}^{(0)} F_{\mu\nu}^{(0)}. \end{aligned}$$

Replacing this into (80) we conclude that the action is the same in both basis, as expected. Note that there is a subtlety here. We used the SW map to render the expression $\sqrt{-g}g^{ae}g^{bf}$ covariant, and in solving the condition (74) we omitted the possible ambiguous terms which solve the homogeneous part of the equation. The reason for this is that in this approach we do not have control over these ambiguities. One should have constructed the SW map for metric g^{bf} with the ambiguous terms included and then calculate the SW map for the expression $\sqrt{-g}g^{ae}g^{bf}$. An easier route might be expressing the metric in terms of vielbeins and constructing the complete SW map for them, but this we leave for future work.

8. Conclusion and outlook

In this paper we used the twist formalism to gain a better understanding of the gauge theory on κ -Minkowski space-time and to resolve certain ambiguities we encounter in our previous analysis [4]. The twist formalism provided us with a naturally defined differential calculus. As a consequence, we obtained uniquely defined derivatives, thus solving one ambiguity. Next, in the twist approach the integral has the trace property, and there is no need to introduce an additional measure function in the integral. This also means that the limit of vanishing deformation parameter a reproduces the undeformed case without the need for additional field redefinitions. One puzzling feature of the gauge field on κ -Minkowski disclosed within the formalism introduced in [6], was that a gauge field is given in terms of the higher order differential operator. This produced "torsion-like" terms in the field strength which were simply omitted in the constructed action. In the twist approach however, the commutation rule of the basis one-forms with functions reproduces the effect of "higher order differential operator" gauge field without producing unwanted terms in the action.

We have shown that the twisting of symmetries, as a way of deforming the algebra of coordinates, is compatible with the local gauge principle. The obstruction we have encountered in the construction of the Hodge-dual field-strength tensor is a manifestation of the fact that the introduction of a noncommutative geometrical structure prevents decoupling of translation and gauge symmetries. We proposed two different possibilities for construction of the dual field strength, both leading to the same effective action in the first order of the deformation parameter.

Although the mixing of space-time and internal symmetries appeared as a problem in our construction, this is in fact one of the most intriguing property of models based on non-trivial algebras of coordinates. One possible way to understand this mixing was offered in the framework of Yang-Mills type matrix models [17]. There it was shown that $U(1)$ part of general $U(N)$ gauge group can be interpreted as induced gravity coupling to the rest ($SU(N)$) gauge degrees of freedom. It would be interesting to see if such an interpretation is possible in our framework, by constructing models with larger gauge groups.

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